A Simplified Statistical Model of Turbulent, Chemically Reacting Shear Flows

PAUL M. CHUNG*
University of Illinois at Chicago Circle, Chicago, Ill.

A turbulence theory has been formulated which describes, in a simplified, tractable manner, the effect of various wavenumber fluctuations on the statistical behavior of the fluid elements containing reactive species. The theory was then employed, as a primary test, to analyze the turbulent Couette flow of a chemically inert fluid. In the analysis, it was shown that the present theory as applied to a chemically inert fluid is self-containing up to the dissipation function. An experimental value of dissipation function for a pipe flow, for lack of the same for a Couette flow, was adapted and used in the analysis. A comparison of the present theoretical results with the available experimental data on Couette flow showed a satisfactory agreement between the two. A simple relationship has been derived between the dissipation function and the other variables of the theory which would render the present theory completely self-containing for the chemically inert single-component flow.

Nomenclature

A = constant of order one
C_1,C_2 = constants of integration defined by Eqs. (34) and (35)
D = diffusion coefficient or diameter of pipe
E = function defined by Eq. (24)
E_1,E_2 = functions defining f_1 and f_2, respectively
$F^{(\alpha)}$ = αth-order distribution function defined by Eq. (3)
f = distribution function of fluid elements
f_1,f_2 = functions defined by Eqs. (19) and (20), respectively
K_i = function defined by Eq. (2)
L = half-width of Couette flow or characteristic length of flow
n = mass fraction of a chemical species
p = pressure
Q = property of fluid elements
Re = Reynolds number, $u_{oc}L/\nu$
t = time
U_i = velocity vector relative to average velocity
u_i = absolute velocity vector
U = relative velocity in x direction
u = absolute velocity in x direction
u_{o1},u_{o2} = functions defining f_1 and f_2, respectively
V,W = relative velocities in y and z directions, respectively
v,w = absolute velocities in y and z directions, respectively
x,y,z = directions shown in Fig. 1
x_i = position vector
Y = y/L (y is measured from a wall, except for plane jet. It is measured from centerline in jet.)
$1/\beta$ = characteristic time of lower wavenumbers
γ = chemical reaction rate constant
δ_{ij} = Kronecker delta
ϵ' = function defined by Eqs. (28)
η = inner variable defined by Eq. (45)
θ = function defined by Eqs. (28)
Λ = characteristic scale of larger eddies
ν = kinematic viscosity
τ = transition probability defined by Eq. (5)
ϕ = function defined by Eqs. (28)
Ψ,ψ = functions defined by Eqs. (36) and (28), respectively

$$\langle Q \rangle = \int_{-\infty}^{\infty} fQdU$$

Subscripts

c = midpoint between two plates in Couette flow
i,j,k,m = Cartesian tensor indices
o = averaged value
w = wall

I. Introduction

TURBULENCE is one of the classical subjects which have been studied for years by many people. Most of the existing works can be divided into two broad categories. The first category is comprised of those studies which analyzed the structure and dynamics of various idealized turbulence fields. The idealizations were necessary because of the extremely complicated nature of the subject. The major contribution of these studies was to give a better description, and therefore a better understanding, of the phenomenon of turbulence itself, rather than to supply the engineers with the solutions of various turbulent-flow problems of practical interest. There are numerous publications belonging to this category.†

In contrast, the other category of works expended its major effort in deriving various relationships which can be employed to study quantitatively the real turbulent-flow problems of engineering interest. By necessity, these works had to rely heavily upon empiricism and phenomenological arguments guided by experimental data. Because of the empiricism, the usefulness of these theories is severely limited. As with the first category, there is an abundance of publications belonging to this phenomenological category.

The problem of "turbulent-wake observables" perhaps has done more to bring these two categories of works together than has any other engineering problem of recent years. When it was first noticed that the radar return from an underdensed plasma comprising a turbulent wake was a function of the mean-square fluctuation of the electron concentration, it became clear that the conventional phenomenological theories were no longer adequate even if one were to accept the inherent empiricism. It became inevitable that at least certain aspects of the statistical approach be brought into the analysis of real flow problems such as the problem of the turbulent wake. The works of Proudian and Feldman,[1] Lin,[2,3] Sutton,[4] and Gibson[4] represent the

---

Received June 14, 1968; revision received April 8, 1969. This work is supported by NASA NGR 14-012-012. The encouragement given by G. Goodwin and P. Nachtscheim of NASA Ames Research Center is greatly appreciated. The development of the first part of the theory was also supported by The Aerospace Corporation, San Bernardino, Calif. This support is hereby acknowledged.

* Professor of Fluid Mechanics, Department of Energy Engineering.

† Because of lack of space, these publications are not listed herein.

earlier endeavor to incorporate portions of the statistical concept into the turbulent-wake analysis in a tractable manner. In the next section we shall be discussing the analyses of Lin[2,3] and Sutton[4] along with a few other pertinent works.

The turbulent wake is in fact only one out of the class of chemically reacting, turbulent shear flow problems that defy solution either by the conventional phenomenological approaches or by the classical statistical turbulence theories. The problems of chemically reacting, turbulent boundary layers and turbulent combustion are a few of the problems also belonging to this class.

In the present paper, a tractable statistical model of chemically reacting, incompressible turbulent shear flows is formulated. Particular effort is made to eliminate the shortcomings of the conventional phenomenological theories pertaining to the turbulent fluctuation, turbulent transport, and the chemical reaction. The theory is then, as a primary test, applied to a chemically inert, turbulent Couette flow. The result is compared with available experimental data. In view of the complexities of the problem, a considerable sacrifice of statistical rigor is made in favor of tractability.

II. Pertinent Previous Works

Several simplified theories have developed since the earlier works[1–5] mentioned in Sec. I, which attempted, as does the present theory, to incorporate various statistical concepts into the real shear flow problems. The works of Harlow and Nakayama[6] and Nee and Kovasznay[7] are the typical ones. The approaches of all these analyses[1–7] are basically different from the present one.‡ Moreover, most of these works are not intended for chemically reacting flows. However, all these works, including the present one, have one general purpose in common: that is to eliminate, as much as possible, the empiricism involved in the conventional phenomenological theories, by the use of a certain statistical concept. We shall in this section present a very brief description of the analyses of Lin,[2,3] Sutton,[4] Harlow and Nakayama,[6] and Nee and Kovasznay.[7]

Lin[2,3] likened the turbulent transport unto the mixing of two classes of fluid elements possessing two diametrically different dissipation characteristics. With this assumption, Lin formulated a set of tractable governing equations for a turbulent wake and solved it. Lin made a valuable contribution in that he was one of the first to pose the possibility of incorporating the statistical concepts into a real flow problem. The major shortcoming of this work[2,3] is that the relationship between the governing equations he developed and those rigorously derived from the Navier-Stokes equation cannot be made clear. Hence, the degree of over-all approximation involved cannot be assessed.

Sutton[4] essentially followed the rather classical approaches previously employed by authors such as Deissler.[8,9] Here, one first considers that the mean profiles are all given a priori. Then one applies the closure to the classical turbulence equations at the lowest order by discarding all correlation terms that are of higher order than the second order. In addition to these procedures, Sutton assumed that the turbulent transport is expressible by the Boussinesq relationship,[10] which says that it is equal to certain turbulent diffusivity times the mean gradient. Furthermore, the dissipation function was also assumed to be known a priori. The equations are then self-containing and amenable to a solution.

Harlow and Nakayama[6] made the classical Reynolds stress equation determinate by introducing several transport flux approximations. These flux approximations were, like Sutton's[4] work, basically patterned after the Boussinesq form. In the approximate expressions, the dissipation scale became an important variable. A transport equation for this dissipation scale was formulated, mainly from a physical argument, and was analyzed along with the Reynold's stress equation.

Nee and Kovasznay[7] formulated a transport equation for a total viscosity, again from a physical argument similar to that employed by Harlow and Nakayama.[6] The total viscosity was defined as the quantity which when multiplied by the local mean velocity gradient yields the total stress due to both molecular and eddy viscosities. This equation, along with the classical turbulent momentum equation, was then analyzed.

III. Formulation of Theory

General Concept

The similarity between the behavior of molecules according to the nonequilibrium kinetic theory and that of fluid elements in turbulence fields has been recognized from the days of Prandtl.[10] Although there exist certain differences between the kinetic theory and the statistical turbulence theory, as was pointed out by Taylor,[10] one can exploit the various fundamental similarities underlying the two statistical phenomena, especially in view of the substantial progress that has been made in kinetic theory during the past decade.

Most of the classical statistical analyses of turbulence begin with the various moment equations derived from the Navier-Stokes equation. Each order averaged quantity such as the Reynold's stress, the triple correlation function, etc. appearing in these moment equations must then be considered as an independent unknown variable. The equations are, hence, intractable for real flow problems unless one is willing to discard certain terms and replace certain other terms by known relationships argued from physical grounds, as Sutton[4] has done.

In order to circumvent this difficulty, we shall develop a governing equation for the lowest-order distribution function of the fluid elements which can be analyzed for the real flow problems. Formulation of turbulence equations in terms of a distribution function is not new in itself, and such formulations derived from the Navier-Stokes equation[11,12] are just as intractable as the more familiar form of turbulence equations.

In the present study, we shall first divorce ourselves from the Navier-Stokes equation, as Lin[2,3] has done. We shall then show that certain similarities between Brownian motion and turbulence can be exploited in such a manner that a modified form of the Fokker-Planck equation can be derived to describe the salient characteristics of turbulent, chemically reacting flows. The consistency of this equation in view of the rigorous statistical turbulence equations is then argued by obtaining various moment equations from this equation and comparing them with the moment equations of the corresponding order derived from the Navier-Stokes equation. For more detailed description of the concept and the equations to be derived below, the reader is referred to a previous report[13] written by this author.

Turbulence Model

One basic factor that is common to all the previous works mentioned in Sec. II is that the turbulent transport phenomena are described in analogy to the molecular transport in a continuum laminar flow.§ Thus, all important diffusions were assumed to take place at rates proportional to the local gradients of the average quantities.

---

‡ The main difference between these analyses and the present study will be discussed in the next section under "Turbulence Model."

§ The only exception to this perhaps is Lin's work.[2,3]

Most of the transport phenomena, however, associated with turbulent motion are due to the larger eddies with long preferred memories. The transport of various properties by these eddies is somewhat analogous to the kinetic-theory description of the molecular transport in a rarefied gas flow where the transport rate is not necessarily determined by the local mean gradient. In addition to the fact that the larger eddies are mainly responsible for the transport, the other observable properties, such as the fluctuation comprising the turbulence energy, are mostly due to these same eddies.

In a physical turbulence field, the various size eddies are intermingled together. A fluid element, therefore, physically belongs to all size eddies. A statistical property of the fluid element is a manifestation of the combined influence of the various size eddies of which the fluid element is a part. The influences of the various size eddies on the fluid element have been discerned through spectral analyses.[10,14] These spectral analyses show the effects of the different wavenumbers representing the different size eddies on the statistical properties of the fluid element.

We assume that the statistical property of a fluid element is entirely due to the lower nonequilibrium wavenumbers, the particular values of which are unimportant, and that the role of the higher equilibrium wavenumbers is to degenerate to a random state and dissipate this property by their interaction with the lower wavenumbers.

Much is known[10,14] about the dynamics of the various wavenumbers (degrees of freedom). In particular, it is known that, when the Reynolds number is sufficiently large, a statistical separation exists between the higher equilibrium wavenumbers and the lower wavenumbers across an inertial subrange. This means, among other things, that there exist two statistically independent characteristic times, a short one for the higher wavenumbers and a long one for the lower wavenumbers, and that these two characteristic times are sufficiently separated from each other when the Reynolds number is large. Furthermore, it is known that the inertial subrange acts as a perfect conductor only in the interaction of the two wavenumber regions.

The preceding description of the dynamics involving the wavenumbers (degrees of freedom) can be readily seen as closely related to Prigogine's description[15] of the generalized Brownian motion. The generalized Brownian motion describes the decay of a nonequilibrium degree of freedom through its interaction with all other degrees of freedom which are in equilibrium. The only condition there is that the characteristic time of the nonequilibrium degree of freedom must be much longer than and separated from that of the equilibrium degrees of freedom.

The characteristic time of the degree of freedom representing the lower wavenumbers (energy containing wave numbers) is known[10,14] to be of the order of $\Lambda/\langle U_k U_k\rangle^{1/2}$ where Λ and $\langle U_k U_k\rangle$ are the characteristic scale of the larger eddies and the average turbulence energy, respectively. The characteristic time for the higher wavenumbers is sufficiently shorter than that of the energy-containing eddies[10,14] when the Reynolds number is large.

Now, the rate at which the instantaneous momentum of a fluid element, which is considered here to be due to the nonequilibrium degree of freedom representing the lower wavenumbers, decays is written according to the generalized Brownian motion[15,16] as

$$\partial u_i/\partial t = -\beta(x_j)(u_i - u_{oi}) + A_i(t) + K_i(x_j,u_i) \qquad (1)^{\P}$$

Equation (1) is Langevin's stochastic equation. The first two terms on the right-hand side of Eq. (1) represent the decay rate of the momentum of the fluid element due to the interaction of the nonequilibrium degree of freedom with the equilibrium degrees of freedom. Physically, this interaction takes place in a most complicated manner over the physical region large enough to accommodate the larger eddies surrounding the fluid element. The important influence of these interactions on the fluid element, however, satisfies the basic criteria of generalized Brownian motion when the Reynolds number is large. We have, therefore, written the linear stochastic equation, Eq. (1), for the decay rate of the influence of the lower wavenumbers (larger eddies) on the fluid element.

The first term on the right-hand side represents the systematic portion, wherein $\beta(x_j)$ represents the characteristic decay rate of the lower wavenumbers, whereas the second term represents the influence of the rapid fluctuations of the equilibrium higher wavenumbers. The value of $\beta(x_j)$, therefore, according to the earlier discussion, is of the order of $\langle U_k U_k\rangle^{1/2}/\Lambda$.

The last term, K_i, is interpreted in the present turbulence model as the force experienced by the fluid element due to the molecular viscosity and the average pressure gradient. The rate at which the momentum of the fluid element changes because of the molecular viscosity is assumed to be given by the value of $\nu\partial^2 u_i/\partial x_m\partial x_m$ observed in that fluid element. The quantity $\nu\partial^2 u_i/\partial x_m\partial x_m$ is considered herein as one of the properties of the fluid element. Thus, we write

$$K_i = \partial^2 u_i/\partial x_m\partial x_m - (1/\rho)\partial p_o/\partial x_i \qquad (2)$$

Distribution Function

We define a distribution function $f(x_i,u_i,t)$ such that $f(x_i,u_i,t)\ d\mathbf{x}\ d\mathbf{u}$** is the occupancy probability of the fluid element in the phase cell $d\mathbf{x}\ d\mathbf{u}$ at time t. Next, we define $n(x_i,u_i,t)$ as the concentration of a transferrable quantity, such as a chemically reactive species, per unit mass of the fluid element containing this quantity. We assume in the present analysis that the fluid is incompressible and that the transferrable quantity is a passive scalar quantity, which means that its effect on turbulence itself is negligible.

With the foregoing definitions of f and n, we now define the αth-order distribution function of the transferrable quantity as

$$F^{(\alpha)}(x_i,u_i,t) = n^{\alpha}(x_i,u_i,t)f(x_i,u_i,t) \qquad (3)$$

where α is a constant which is equal to or greater than zero. The probability average of α power of n, $\langle n^{\alpha}\rangle$, is then

$$\langle n^{\alpha}\rangle = \int F^{(\alpha)}d\mathbf{u} \qquad (4)\dagger\dagger$$

which, with a suitable ergodic argument for a steady-state system[14,17] (stationary random system with respect to time), is the same as the time average of n^{α}.

Transition Probability

We have previously exploited the analogy between the present turbulence model and Brownian motion. Brownian motion is a Markov process general treatment of which can be found elsewhere.[15,16] The portions of the development which can be found in these references are slighted in the following analysis.

Let Δt be a time interval that lies between the two characteristic times of the lower and the higher wavenumbers. Such a time interval exists because the statistical separation of the two wavenumber regions has been considered to exist. We then let $\tau(x_j - u_j\Delta t, u_j - \Delta u_j; \Delta u_j)$ be the transition probability of the fluid element containing the chemical species between the two phase points $(x_j - u_j\Delta t, u_j - \Delta u_j)$

¶ We employ the Cartesian tensor notation.

** $d\mathbf{x}\ d\mathbf{u} = dx_1dx_2dx_3du_1du_2du_3$.

†† $\int d\mathbf{u} = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \int_{-\infty}^{\infty} du_3$.

and (x_j,u_j) connected by the path of the fluid element. This transition probability is now determined from the stochastic solution of Eq. (1) to satisfy the boundary condition that, as $1/\beta \rightarrow 0$, $F^{(1)}$ must approach Maxwellian at the average turbulence energy $\frac{1}{3}\langle U_k U_k\rangle$. In a steady-state flow system, $1/\beta \rightarrow 0$ implies that the characteristic turbulence decay time is much shorter than other characteristic times of the system.

In the present theory, in which the turbulence energy is contained in the lower wavenumbers (larger eddies) only, the boundary condition of approaching Maxwellian at the average turbulence energy implies the following: The effect of the interaction between the nonequilibrium lower wavenumbers and the equilibrium higher wavenumbers represented by the first two terms on the right-hand side of Eq. (1) is to degenerate the preferred memories of the fluid element, which are due to the lower wavenumbers, toward the random state at the constant average energy of the fluid element. The last term of Eq. (1) then contains the dissipation of the momentum out of the fluid element by the molecular viscosity.

The stochastic solution of Eq. (1) is obtained in a manner similar to that obtained in Ref. 16, although the present boundary condition is a bit different from that employed there, as

$$\tau(u_i;\Delta u_i) = \frac{1}{(\frac{4}{3}\pi\langle U_k U_k\rangle\beta\Delta t)^{3/2}} \exp - \{[\Delta u_i + (\beta u_i - K_i)\Delta t]^2/(\tfrac{4}{3}\langle U_k U_k\rangle\beta\Delta t)\} \tag{5}$$

Governing Equation for $F^{(1)}$

The governing equation for the distribution function $F^{(1)}$ is formulated herein. This equation is the Fokker-Planck equation adapted and modified for the present purpose. By setting $n = 1$, we then readily obtain the governing equation for $f(= F^{(0)})$ as the degenerate case of the equation for $F^{(1)}$.

Since we are considering only one chemically reactive species mixed with an incompressible chemically inert fluid, the only appropriate chemical reaction is that given below:

$$(\partial n/\partial t)_{\text{chem}} = \gamma n^{\alpha} \tag{6}$$

where γ is a rate constant which may be either positive or negative. We assume that the molecular diffusion of n into and out of the fluid element takes place in accordance with the value of $D\partial^2 n/\partial x_m \partial x_m$ observed in that fluid element.

The governing integral equation for $F^{(1)}$ is now constructed as

$$F^{(1)}(x_i,u_i,t + \Delta t) = \int_{-\infty}^{\infty} \Big[F^{(1)}(x_i - u_i\Delta t,u_i - \Delta u_i,t) + \gamma F^{(\alpha)}(x_i - u_i\Delta t,u_i - \Delta u_i,t)\cdot\Delta t + D\left(\frac{\partial^2 n}{\partial x_m \partial x_m}\right)_{x_i - u_i\Delta t,u_i - \Delta u_i,t}\cdot f(x_i - u_i\Delta t,u_i - \Delta u_i,t)\cdot\Delta t \Big]\cdot\tau(x_i - u_i\Delta t,u_i - \Delta u_i;\Delta u_i)d(\Delta\mathbf{u}) \tag{7}$$

In Eq. (7) the left-hand side is, of course, the probability of finding the chemical species at the phase point (x_i,u_i) at time $t + \Delta t$. The quantity within the square bracket on the right represents the probability of finding the species at the neighboring phase point at time t, plus the probabilities that the species concentration contained in the fluid element will be changed by the chemical reaction and the molecular diffusion during the time interval Δt that the fluid element takes for the transition between the two phase points. On the right-hand side of Eq. (7), then, the aforementioned quantity is multiplied by the transition probability and is integrated over all the neighboring points accessible to the point (x_i,u_i) during the time Δt.

Equation (7) is transformed into a differential equation by expanding the various functions in Taylor series and by discarding the terms of order $(\Delta t)^2$ and higher. Then, after a considerable manipulation,[13] there results, with the use of Eqs. (2) and (5),

$$u_j\frac{\partial F^{(1)}}{\partial x_j} + \frac{\partial}{\partial u_j}\left[F^{(1)}\left(\nu\frac{\partial^2 u_j}{\partial x_m \partial x_m} - \frac{1}{\rho}\frac{\partial p_o}{\partial x_j}\right)\right] = \beta\left[\frac{\partial}{\partial U_j}(F^{(1)}U_j) + \frac{\langle U_k U_k\rangle}{3}\frac{\partial^2 F^{(1)}}{\partial U_k \partial U_k}\right] + Df\frac{\partial^2 n}{\partial x_m \partial x_m} + \gamma F^{(\alpha)} \tag{8}$$

Equation (8) is the fundamental equation for the present purpose which governs the distribution function $F^{(1)}$, for steady state.

It is readily seen that, as $1/\beta \rightarrow 0$, Eq. (8) gives the following limiting solution which was to be expected from the boundary condition applied on Eq. (1):

$$F^{(1)}(x_i,u_i) = \frac{\langle n\rangle}{[\frac{2}{3}\pi\langle U_k U_k\rangle]^{3/2}} \exp - \left[\frac{U_k U_k}{\frac{2}{3}\langle U_k U_k\rangle}\right] \tag{9}$$

It can be shown,[13] in a manner similar to the Chapman-Enskog expansion in kinetic theory,[18] that, as $1/\beta$ is slightly increased from zero and as $F^{(1)}$ is perturbed slightly from the Maxwellian distribution of Eq. (9), the transport rates of momentum and chemical species become proportional to the mean velocity and concentration gradients, respectively. In this near-equilibrium (near-isotropic) limit, therefore, the Boussinesq relationship is valid. In a typical shear flow, however, $1/\beta$ is not sufficiently small to insure the validity of the Boussinesq relationship.

Equation (8) is not determinate because the quantities $(\partial^2 u_i/\partial x_m \partial x_m)$ and $(\partial^2 n/\partial x_m \partial x_m)$ are not known. Since these quantities have been assumed to be properties of the fluid elements, either they must be considered as additional independent variables for the distribution function or one must be able to relate them to other properties which have been already described.

In the subsequent analysis of chemically inert Couette flow, Eq. (8) will be analyzed by an approximate method which utilizes the moment equations derived from that equation. Out of those terms appearing in the moment equations which are generated from the term $(\partial^2 u_i/\partial x_m \partial x_m)$, only one term is important and is unknown‡‡: the dissipation function. We shall analyze the problem first by employing the experimentally measured value of the dissipation function. Later, a theoretical approximation of the dissipation function will be given in the light of the solution obtained.

Moment Equations

The generalized moment equation will be first derived from Eq. (8). Several particular order moment equations will then be deduced from the generalized moment equation. The reason for deriving these moment equations is twofold. The first is that the particular order moment equations derived will be compared with the corresponding moment equations classically derived from the Navier-Stokes equation. From this comparison, we shall be able to establish a certain consistency of the present model in the light of the rigorous classical approach. Secondly, as we mentioned in the preceding subsection, we shall be utilizing these moment equations for the solution of the Couette flow.

We let $Q(x_i,U_i)$ be a generalized property function and multiply Eq. (8) through by this function. Then, after

---

‡‡ In that problem, $n = 1$ and $\partial^2 n/\partial x_m \partial x_m = 0$.

some manipulation similar to that employed in deriving the Enskog's generalized moment equation from the Boltzmann equation,[18] we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial x_k}\int (u_{ok} + U_k)QF^{(1)}d\mathbf{U} - \int (u_{ok} + U_k)\frac{\partial Q}{\partial x_k}\,d\mathbf{U} + \\
&\qquad \int \frac{\partial Q}{\partial U_m}\left[(u_{ok} + U_k)\frac{\partial u_{om}}{\partial x_k}\right]F^{(1)}d\mathbf{U} - \\
&\qquad \int \frac{\partial Q}{\partial U_k}\left[\nu\frac{\partial^2(u_{ok} + U_k)}{\partial x_m\partial x_m} - \frac{1}{\rho}\frac{\partial p_o}{\partial x_k}\right]F^{(1)}d\mathbf{U} = \\
&-\beta\left[\int U_k\frac{\partial Q}{\partial U_k}F^{(1)}\,d\mathbf{U} + \frac{\langle U_kU_k\rangle}{3}\int\frac{\partial Q}{\partial U_k}\frac{\partial F^{(1)}}{\partial U_k}\,d\mathbf{U}\right] + \\
&\qquad \gamma\int F^{(\alpha)}Qd\mathbf{U} + D\int fQ\frac{\partial^2 n}{\partial x_m\partial x_m}\,d\mathbf{U} \qquad (10)
\end{aligned}
$$

Now, Eq. (10) can be first reduced to the generalized moment equation for $F^{(0)} = f$ by setting $n = 1$. Then, as we substitute $Q = 1$ and $Q = U_i$ successively in Eq. (10), there result continuity and momentum equations that are identical to the corresponding equations derived from the Navier-Stokes equation. Next, we derive the following stress tensor equation by substituting $Q = U_iU_j$ in Eq. (10):

$$
\begin{aligned}
&\langle U_iU_k\rangle\frac{\partial u_{oj}}{\partial x_k} + \langle U_jU_k\rangle\frac{\partial u_{oi}}{\partial x_k} + u_{ok}\frac{\partial}{\partial x_k}\langle U_iU_j\rangle = \\
&\quad -\frac{\partial}{\partial x_k}\langle U_iU_jU_k\rangle - 2\beta\left[\langle U_iU_j\rangle - \frac{1}{3}\delta_{ij}\langle U_kU_k\rangle\right] + \\
&\qquad \nu\left[\frac{\partial^2\langle U_iU_j\rangle}{\partial x_m\partial x_m} - 2\left\langle\frac{\partial U_i}{\partial x_m}\frac{\partial U_j}{\partial x_m}\right\rangle\right] \qquad (11)\S\S
\end{aligned}
$$

where we made use of the identity

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(U_i\frac{\partial f}{\partial U_j} + U_j\frac{\partial f}{\partial U_i}\right)d\mathbf{U} = \\
&\qquad -\int_{-\infty}^{\infty} f\left(\frac{\partial U_i}{\partial U_j} + \frac{\partial U_j}{\partial U_i}\right)d\mathbf{U} = -2\delta_{ij} \qquad (12)
\end{aligned}
$$

The turbulence energy equation is derived by a contraction of Eq. (11) as

$$
\begin{aligned}
&u_{ok}\frac{\partial}{\partial x_k}\langle U_iU_i\rangle + 2\langle U_iU_k\rangle\frac{\partial u_{oi}}{\partial x_k} = -\frac{\partial}{\partial x_k}\langle U_kU_iU_i\rangle + \\
&\qquad \nu\left[\frac{\partial^2\langle U_iU_i\rangle}{\partial x_m\partial x_m} - 2\left\langle\frac{\partial U_i}{\partial x_m}\frac{\partial U_i}{\partial x_m}\right\rangle\right] \qquad (13)
\end{aligned}
$$

Similar moment equations for the chemical species can be derived from Eq. (10). However, for lack of space here we shall include them in a future paper.

It has been already stated that the continuity and momentum equations obtained from Eq. (10) by substituting $Q = 1$ and $Q = U_i$, respectively, are identical to the corresponding equations derived from the Navier-Stokes equation. A comparison of Eq. (11) with the corresponding stress equation derived from the Navier-Stokes equation¶¶ shows that the two are identical if

$$
\begin{aligned}
&2\beta\left[\langle U_iU_j\rangle - \frac{1}{3}\delta_{ij}\langle U_kU_k\rangle\right] = \\
&\frac{1}{\rho}\left[\frac{\partial}{\partial x_i}\langle pU_j\rangle + \frac{\partial}{\partial x_j}\langle pU_i\rangle\right] - \frac{1}{\rho}\left\langle p\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)\right\rangle \qquad (14)
\end{aligned}
$$

A contraction of Eq. (14) gives

$$(\partial/\partial x_k)\langle pU_k\rangle = 0 \qquad (15)$$

Implications of Eqs. (14) and (15) are as follows: First, the present theory neglects the first term on the right-hand side of Eq. (14) as compared with the second term, and second, the present equation when compared to the classical stress equation says that

$$
\begin{aligned}
&2\beta\left[\langle U_iU_j\rangle - \frac{1}{3}\delta_{ij}\langle U_kU_k\rangle\right] = \\
&\qquad -\frac{1}{\rho}\left\langle p\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)\right\rangle \qquad (16)
\end{aligned}
$$

Equation (16) is identical to the relationship between the stress and the pressure-velocity-gradient correlation function derived by Rotta[10] if we let

$$\beta = (A/2\Lambda)[\langle U_kU_k\rangle]^{1/2} \qquad (17)$$

Equation (17) is consistent with the meaning and the value which we originally attached to β of Eq. (1) in the formulation of the turbulence model. Furthermore, the expression on the left-hand side of Eq. (16) resulted from the present description of the decay of the preferred memories of the larger eddies toward the randomized state. Reference 10 gives a rather detailed physical explanation showing that the right-hand side of Eq. (16) is the term which describes precisely the same decay of the preferred memories of the eddies in the classical turbulence stress equation; hence Eq. (16) is essentially a physical identity.

Now, we have shown that the present moment equation, Eq. (10), is consistent with the classically derived moment equations, at least up through the stress equations, except for the approximation that it neglects the first term as compared with the second term on the right-hand side of Eq. (14).

A similar comparison can be made between the moment equations derived from Eq. (10) including the variable n and the classically derived moment equations for the chemical species n. This comparison is given in Ref. 13 and is not included here for the lack of space.

This concludes the formulation of the general theory. We shall now apply the theory, as a first test, to the simplest flow configuration of Couette flow.

IV. Couette Flow of a Chemically Inert Fluid

Formulation

We consider a turbulent Couette flow of a chemically inert fluid, as shown in Fig. 1. Because of the reason given following Eq. (9), we shall analyze Eq. (8) for this problem by a moment method that has been previously employed in solving the Boltzmann equation for highly nonequilibrium (from the viewpoint of kinetic theory) rarefied gas flows or shock structures. Out of the numerous methods available, we shall employ the moment method first used by Liu and Lees[19] in the solution of a rarefied Couette flow. It is recognized at the outset that this is not the one best suited for the present turbulent-flow problem. This method is chosen, however, because of its simplicity. Since the detailed discussion of the method is given in Ref. 19, we shall in the following apply the method to the present problem without elaboration.

We first approximate the distribution function f by the following two-stream half Maxwellian functions:

$$f = f_1 + f_2 \qquad (18)$$

$$f_1 = \frac{1}{(\frac{2}{3}\pi E_1)^{3/2}}\exp-\left[\frac{(u - u_{o1})^2 + V^2 + W^2}{2E_1/3}\right], \text{ for } V > 0 \qquad (19)$$

$$f_2 = \frac{1}{(\frac{2}{3}\pi E_2)^{3/2}}\exp-\left[\frac{(u - u_{o2})^2 + V^2 + EW^2}{2E_2/3}\right], \text{ for } V < 0 \qquad (20)$$

and f_1 and f_2 are zero for $V < 0$ and $V > 0$, respectively.

---

§§ Here we assumed that the present probability average is the same as the time average and, hence, we replaced $\langle\partial^2 U_iU_j/\partial x_m\partial x_m\rangle$ by $\partial^2\langle U_iU_j\rangle/\partial x_m\partial x_m$.

¶¶ See, for instance, Eq. (4-3) of Ref. 10.

In Eqs. (19) and (20), E_1, E_2, u_{o1}, and u_{o2} are the unknown functions of y to be determined.

With the preceding distribution function, the salient mean quantities are defined in terms of the functions E_1, E_2, u_{o1}, and u_{o2} as

$$u_o = \langle u\rangle = \int_{-\infty}^{\infty} dU \int_{0}^{\infty} dV \int_{-\infty}^{\infty} dW f_1 u + \int_{-\infty}^{\infty} dU \int_{-\infty}^{0} dV \int_{-\infty}^{\infty} dW f_2 u = \frac{u_{o1} + u_{o2}}{2} \qquad (21)$$

$$\langle UV\rangle = \frac{1}{(6\pi)^{1/2}} \left[-\left(\frac{u_{o1} + u_{o2}}{2}\right)(E_1^{1/2} - E_2^{1/2}) + (u_{o1}E_1^{1/2} - u_{o2}E_2^{1/2}) \right] \qquad (22)$$

$$\langle U_kU_k\rangle = \left(\frac{u_{o1} - u_{o2}}{2}\right)^2 + \left(\frac{1}{2} - \frac{1}{6\pi}\right)(E_1 + E_2) + \frac{1}{3\pi}(E_1E_2)^{1/2} \qquad (23)$$

We now employ the first four moment equations derived in the preceding section from the generalized moment equation, Eq. (10). These are the continuity equation, the momentum equation, and Eqs. (11) and (13). These equations are first simplified by recognizing the fact that $v_o = w_o = \partial/\partial x = \partial/\partial z = 0$ for Couette flow. The various average quantities of the equations are then evaluated in terms of E_1, E_2, u_{o1}, and u_{o2} by the use of Eqs. (18–20). The resulting continuity equation is immediately integrated with the boundary condition of $v = 0$ at the wall to give

$$E_1 = E_2 = E \qquad (24)$$

The remaining three equations are now manipulated and nondimensionalized to give

$$\frac{1}{Re}\frac{d^2\phi}{dY^2} = \left(\frac{2}{3\pi}\right)^{1/2} \frac{d(\psi\theta)}{dY} \qquad (25)$$

$$\frac{1}{Re}\left(\frac{6}{\pi}\right)^{1/2} \frac{d^2(\psi\theta)}{dY^2} = 2\left(\frac{6}{\pi}\right)^{1/2}\left(\frac{\psi^2}{4} + \theta^2\right)^{1/2} \times (\psi\theta) + \theta^2\frac{d\phi}{dY} \qquad (26)$$

$$\frac{1}{Re}\left(\frac{3\pi}{8}\right)^{1/2} \frac{d^2(\psi^2 + 4\theta^2)}{dY^2} = (\psi\theta)\frac{d\phi}{dY} + 2(6\pi)^{1/2}\frac{L}{u_{oc}^3}\epsilon' \qquad (27)$$

where

$$\begin{gathered} \phi = (u_{o1} + u_{o2})/u_{oc}, \quad \psi = (u_{o1} - u_{o2})/u_{oc} \\ \theta = E^{1/2}/u_{oc}, \quad Y = y/L, \quad Re = u_{oc}L/\nu \\ \epsilon' = \nu\langle(\partial U_k/\partial x_m)\partial U_k/\partial x_m\rangle \end{gathered} \qquad (28)$$

In deriving (26), it was assumed that $\nu\langle(\partial U/\partial x_m)\partial V/\partial x_m\rangle \ll \beta\langle UV\rangle$. This assumption implies that the effect of molecular viscosity in degenerating the preferred memories of the larger eddies is negligible as compared with that of the momentum interaction between the eddies.

Also, in Eq. (26), the expression of Eq. (17) was employed for β with $A = 1$ and $\Lambda = \frac{1}{2}L$. This particular value of Λ was chosen because of the following reason: The quantity Λ is the characteristic scale of the larger eddies, which is probably of order y measured from the wall, reaching the maximum value of L at the midpoint between the two plates. For simplicity, it was decided to set Λ equal to an average value of $L/2$ for the entire flow region.

The only unspecified quantity in Eqs. (25–27) is the dissipation function* ϵ' appearing in Eq. (27). As has been already stated following Eq. (9), we shall employ an experimentally measured value of ϵ' for pipe flow in the solution of Eqs. (25–27). At the end of the present paper, however, we shall advance an approximate theory which predicts the dissipation function.

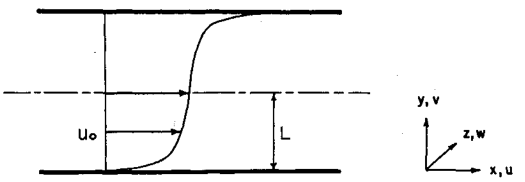

Fig. 1 Couette flow configuration.

The dissipation function measured by Laufer[10] for pipe flow is used because no published dissipation function is available for Couette flow, and also because pipe flow is perhaps more similar to Couette flow than any other flow configuration. The following modification, however, was first made on the value of the dissipation function measured in pipe flow so that it will more closely describe the dissipation in Couette flow.

In Couette flow, the mean velocity at the midpoint u_{oc} is the average mean velocity of the flowfield between the two plates. The mean velocity at the center of pipe flow, on the other hand, is the maximum velocity but is not the average mean. We assumed that the dissipation in Couette flow normalized with u_{oc} and L is equal to that in pipe flow normalized with the average mean velocity u_{ave} and $D/2$. Relationships were needed between the friction velocity, maximum velocity, and average mean velocity of the pipe flow in order to translate the dissipation data of Laufer. These relationships are found in Ref. 20, and the eighth-power mean velocity profile for pipe flow was employed in utilizing these relationships.

According to Ref. 10, out of the two Reynolds numbers at which the measurement of the dissipation function was made, the most reliable measurement is that which was made at the Reynolds number, based on the maximum velocity and the pipe radius, of 2.5×10^4. At the same time, the most complete experimental values of mean velocity profile for Couette flow are presented in Ref. 21 at $Re = 10^4$, with which we would subsequently compare the present theoretical results. It was decided, therefore, to solve the governing equations, Eqs. (25–27), for $Re = 10^4$, with an interpolated pipe dissipation value.

In accordance with the discussions presented in the preceding two paragraphs, we obtained the following expression for the dissipation function for Couette flow at $Re = 10^4$:

$$\epsilon' L/u_{oc}^3 = \begin{cases} 0.0004/Y, & \text{for } 0 < Y \leq 1 \\ 0.0004/(2 - Y), & \text{for } 1 \leq Y < 2 \end{cases} \qquad (29)$$

Equation (29) was chosen because it is simple, yet it approximates the experimental value closely for all Y except for $Y \to 0$ (and $Y \to 2$) and $Y = 1$. On a physical ground, it is obvious that ϵ' should decrease near the wall and should become zero at the wall. Also, at $Y = 1$, $d\epsilon'/dY$ must be zero from the symmetry condition. The fact that Eq. (29) fails to describe the dissipation correctly at these isolated regions is not critical in the present analysis, and these points will be discussed further subsequently.

The boundary conditions for Eqs. (25–27) are as follows. At $Y = 0$,

$$\phi = \psi\theta = \psi^2 = 0 \qquad (30)$$

and at $Y = 2$,

$$\phi = 4, \quad \psi\theta = \psi^2 = 0 \qquad (31)$$

An inspection of Eqs. (25–27) shows that these equations exhibit a singular behavior at large values of Re. We shall

---

* To be precise, ϵ' is not exactly equal to dissipation. It is, however, sufficiently close to it, and ϵ' is usually measured to represent the dissipation.[10]

therefore exploit this singular behavior and solve the governing equations by the method† of "inner and outer" expansions in the following.

Manipulation of Equations

The relative orders of magnitude of the various terms comprising the governing equations can be made clear through the following manipulations. Equation (25) is first integrated once. The resulting equation is then combined with Eqs. (26) and (27) to give

$$\frac{1}{Re}\left[\frac{d^3\phi}{dY^3} - \frac{\pi}{2}\frac{d\phi}{dY}\left(\frac{d\phi}{dY} - C_1\right)^2 \Big/ \psi^2\right] - 2\left[\frac{\psi^2}{4} + \frac{3\pi}{2}\left(\frac{1}{Re}\right)^2\left(\frac{d\phi}{dY} - C_1\right)^2 \Big/ \psi^2\right]^{1/2} \times \left(\frac{d\phi}{dY} - C_1\right) = 0 \quad (32)$$

$$\frac{1}{Re}\frac{d^2}{dY^2}\left[\psi^2 + 6\pi\left(\frac{1}{Re}\right)^2\left(\frac{d\phi}{dY} - C_1\right)^2 \Big/ \psi^2\right] - 2\left(\frac{1}{Re}\right)\frac{d\phi}{dY}\left(\frac{d\phi}{dY} - C_1\right) - 8\left(\frac{\epsilon' L}{u_{oc}{}^3}\right) = 0 \quad (33)$$

where C_1 is the constant of integration of Eq. (25). We have now reduced Eqs. (25–27) to the two equations, Eqs. (32) and (33). The coupled equations (32) and (33) govern ϕ and ψ^2. θ is then given by Eq. (25). Equation (25) converts the boundary condition $\psi\theta = 0$ of Eqs. (30) and (31) into the following condition at $Y = 0$ and $Y = 2$:

$$(d\phi/dY)_w = C_1 \quad (34)$$

It is known from the experimental data[21,23] on surface skin friction that $(d\phi/dY)_w$ is of much greater order of magnitude than one, and, also, it is related to Reynolds number. We therefore first let

$$C_1 = (d\phi/dY)_w = C_2/(1/Re)^q \quad (35)$$

where C_2 is a constant of order one. We also let

$$\psi^2 = (1/Re)^m \Psi \quad (36)$$

with Ψ considered as order one. The exponents m and q, which are positive, will be determined subsequently.

Equations (35) and (36) transform Eqs. (32) and (33), along with Eq. (25), to

$$\left(\frac{1}{Re}\right)^{m+2q}\frac{d^3\phi}{dY^3} - \frac{\pi}{2}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right]^2\frac{d\phi}{dY}\Big/\Psi - 2\left(\frac{1}{Re}\right)^{(3m/2)+q-1}\left\{\frac{\Psi}{4} + \frac{3\pi}{2\Psi}\left(\frac{1}{Re}\right)^{2(1-m-q)}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right]^2\right\}^{1/2}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right] = 0 \quad (37)$$

$$\left(\frac{1}{Re}\right)^{m+q}\frac{d^2}{dY^2}\left\{\Psi + 6\pi\left(\frac{1}{Re}\right)^{2(1-m-q)}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right]^2\Big/\Psi\right\} - 2\frac{d\phi}{dY}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right] - 8\left(\frac{1}{Re}\right)^{q-1}\left(\frac{\epsilon' L}{u_{oc}{}^3}\right) = 0 \quad (38)$$

$$\Psi^{1/2}\theta = \left(\frac{3\pi}{2}\right)^{1/2}\left(\frac{1}{Re}\right)^{1-(m/2)-q}\left[\left(\frac{1}{Re}\right)^q\frac{d\phi}{dY} - C_2\right] \quad (39)$$

Boundary conditions for Eqs. (37–39) are those given by Eqs. (30, 31, and 34).

The coefficients of the highest-order differential terms of Eqs. (37–39) are very small. This indicates that there exists an "inner" region[22,24] near the wall. These highest-order terms describe the laminar transport, and this inner region, only within which the laminar transport is important, would correspond to the conventional "laminar sublayer." The method of the "inner and outer" expansions[22,24] (singular perturbation) is well suited for the solution of such a problem. It is pointed out here at the outset, however, that Eq. (38) with $\epsilon' L/u_{oc}{}^3$ given by Eq. (29) is valid for $Re = 10^4$ only. Hence, the full use of the method, resulting, for instance, in the universal values for m and q, is not possible here since such analysis requires the functional relationship between the dissipation function and the Reynolds number.

In the following analysis by the singular perturbation, Re is simply a large number equal to 10^4, and the values of m and q are chosen so as only to render the salient terms of the inner and the outer regions order one for $Re = 10^4$.

Outer Solution: Fully Turbulent Region

After a study of the various terms of Eqs. (37–39), it was decided to let $m = \frac{1}{4}$ and $q = \frac{1}{2}$. We shall, in the outer region, consistently neglect the terms of order $(1/Re)^{1/2}$ and higher. Up to $0(1/Re)^{1/2}$, Eqs. (37–39) become

$$\frac{\pi}{2}\frac{C_2}{\Psi}\frac{d\phi}{dY} = 2(10)^{1/2}\left[\frac{\Psi}{4} + \frac{3\pi}{2}(10)^{-2}C_2{}^2\Big/\Psi\right]^{1/2} \quad (40)$$

$$\frac{d\phi}{dY} = \begin{cases}(0.33/2C_2)1/Y \text{ for } 0 < Y \leqq 1\\ (0.33/2C_2)1/(2 - Y) \text{ for } 1 \leqq Y < 2\end{cases} \quad (41)$$

$$\Psi^{1/2}\theta = -(3\pi/2)^{1/2}(10)^{-3/2}C_2 \quad (42)$$

From the symmetry which exists between $Y < 1$ and $Y > 1$, the two boundary conditions on ϕ given in Eqs. (30) and (31) imply that $\phi(1) = 2$. Equation (41) is now integrated and, after applying the boundary condition $\phi(1) = 2$, there results

$$\phi = \begin{cases}(0.33/2C_2)\ln Y + 2 \quad \text{for } Y \leqq 1\\ (0.33/2C_2)\ln(2 - Y) + 2 \text{ for } Y \geqq 1\end{cases} \quad (43)$$

Equation (41) is then substituted into Eq. (40) for $d\phi/dY$ and there results for Ψ

$$\begin{aligned}&\Psi^3 + 6\pi(10^{-2})C_2{}^2\Psi - \frac{(0.33\pi)^2}{160}\frac{1}{Y^2} = 0 \qquad \text{for } Y \leqq 1\\ &\Psi^3 + 6\pi(10^{-2})C_2{}^2\Psi - \frac{(0.33\pi)^2}{160}\frac{1}{(2 - Y)^2} = 0 \text{ for } Y \geqq 1\end{aligned} \quad (44)$$

Standard solution for linear cubic equations is, of course, available, and, therefore, Ψ described by Eqs. (44) is considered to have been obtained. With ϕ and Ψ known, Eq. (42) immediately gives θ.

Equations (43) and (44) show that ϕ is antisymmetric and Ψ is symmetric about $Y = 1$, as was expected. In addition to this, we should expect that $d^2\phi/dY^2 = d\Psi/dY = 0$ at $Y = 1$ on physical grounds, which are not satisfied by Eqs. (43) and (44). One can readily see, however, that this discrepancy is precisely due to the fact that we approximated the dissipation function by Eq. (29), which, although symmetric, does not give the correct physical behavior of $d(\epsilon' L/u_{oc}{}^3)/dY = 0$ at $Y = 1$. This point was explained earlier, following Eq. (29). However, the values of $d^2\phi/dY^2$ and $d\Psi/dY$ at $Y = 1$ given by Eqs. (43) and (44) are negligibly small.

The outer solution has now been obtained up to the unknown constant C_2. This constant, which represents the surface shear [see Eq. (35)], is determined through a matching with the inner solution which is described below.

---

† See Ref. 22 for the general method.

Inner Solution: Laminar Viscous Region

It was mentioned following Eq. (29) that this equation, approximating the dissipation, should fail near the wall. In fact, the dissipation should decrease toward zero through the inner region. In the inner region, the laminar, viscous phenomena predominate over the turbulent phenomena including the dissipation. It is assumed in the following that the dissipation term, whose correct value is not known, is of smaller order of magnitude than other terms of the governing equations in the inner region.

Now, for near $Y = 0$, we stretch the inner region by defining an inner independent variable by

$$\eta = Y/(1/Re)^{1/2} \qquad (45)$$

Equations (37) and (38) become up to $0(1/Re)^{1/2}$, with the use of Eq. (45), as

$$\frac{d^3\phi}{d\eta^3} - 10^{-1/2}\Psi^{1/2}\left(\frac{d\phi}{d\eta} - C_2\right) - 5\pi\frac{d\phi}{d\eta}\left(\frac{d\phi}{d\eta} - C_2\right)^2 \Big/ \Psi = 0 \qquad (46)$$

$$\frac{d^2\Psi}{d\eta^2} = 20\frac{d\phi}{d\eta}\left(\frac{d\phi}{d\eta} - C_2\right) \qquad (47)$$

Equation (39) is unchanged except that $(1/Re)^q d\phi/dY$ becomes $d\phi/d\eta$.

Boundary conditions for Eqs. (45–47) at the wall are those given by Eqs. (30) and (35). The other set of boundary conditions are that ϕ and Ψ should asymptotically match to those given by the outer solutions, Eqs. (43) and (44), respectively, as $\eta \to \infty$. This point will be discussed further in the following.

Matching

The matching near $Y = 0$ will be discussed herein. The matching near the other surface, $Y = 2$, is the same as that near $Y = 0$.

We essentially have the outer solution in closed form, up to the unspecified constant C_2, which is given in Eqs. (43) and (44). The inner solution, on the other hand, must be obtained by integrating Eqs. (46) and (47) numerically. The boundary conditions for Eqs. (46) and (47) are Eqs. (30) at $\eta = 0$, and that ϕ and Ψ approach those given by the outer solutions Eqs. (43) and (44), respectively, as $\eta \to \infty$. These boundary conditions at $\eta \to \infty$ for the inner equations insure the matching between the two regions.

C_2 is an unspecified constant appearing both in the outer solutions and in the inner equations. There is a particular value of C_2 which permits the asymptotic matching of the two regions and, at the same time, satisfies the relationship Eq. (35). This matching solution is obtained by iterating on C_2.

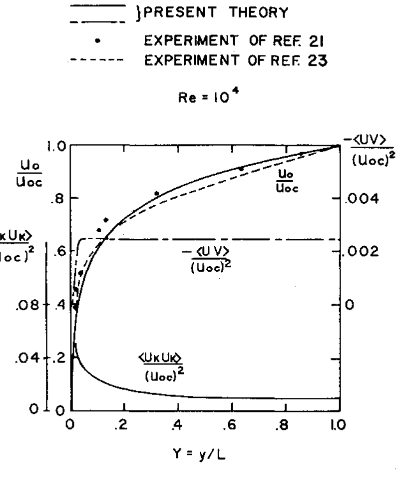

Fig. 2 Couette flow solution and experimental results.

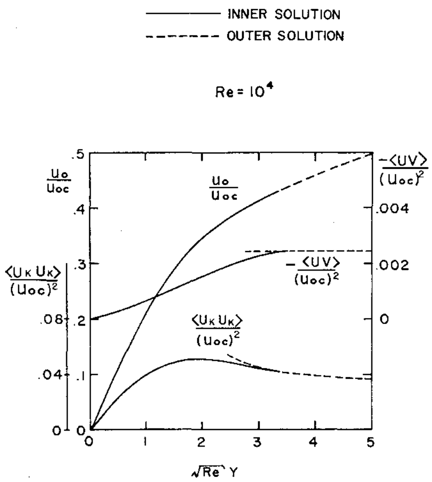

Fig. 3 Inner solution for Couette flow.

Couette Flow Results

From the solutions of Eqs. (25–27) described in the preceding subsections, the profiles of the mean velocity, Reynolds stress, and turbulence energy have been computed by the use of Eqs. (21–23). These quantities are plotted in Fig. 2. The stretched inner profile (laminar sublayer) is shown in Fig. 3.

The available experimental data for Couette flow are summarized in a paper by Robertson.[21] All the published measurements were on the mean velocity profile and the surface shear, and no experimental data are available for the Reynolds stress and the turbulence energy.

Figure 2 shows that the velocity profile obtained in the present analysis compares very satisfactorily with the experimental values. The coefficient of skin friction C_f was computed from the present solution for $Re = 10^4$ as 0.005. This value is within a percent or two of that measured in the experiment of Robertson.[21] Since the approximate dissipation function deduced from the experimental data of pipe flow was employed in the present analysis, this close agreement of C_f with that of Robertson is rather fortuitous.

V. Approximation of Dissipation Function

An approximate expression for the dissipation function applicable to the shear flows in general will be derived herein. As it was discussed in Sec. III under "Turbulence Model," much is known about the dynamics of the various size eddies or the various wavenumbers (see, for instance, Refs. 10 and 14). Although most of the dynamical analyses were made for idealized turbulence fields, such as homogeneous and isotropic fields, many of the general theories and descriptions of turbulence fields given by these analyses apply, in principle, to the shear flows of present interest. For instance, the statistical separation of the larger and the smaller eddies for high Reynolds numbers, and the existence of an inertial subrange, are expected in the shear flows as well as in the idealized turbulence fields.[14]

It is the aforementioned general statistical separation of the wavenumbers upon which the present turbulence model has been built. However, it is not necessary in the formulation of the present turbulence model that the "universal equilibrium theory"[14] for homogeneous and isotropic turbulence fields, which led to the Kolmogoroff spectrum, should hold

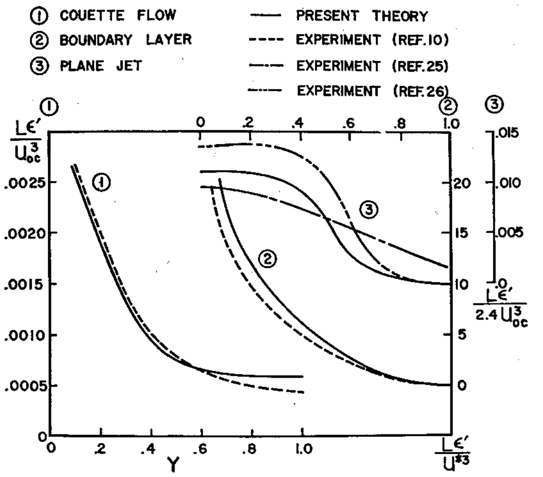

Fig. 4 Dissipation functions (u^* = friction velocity).

in its entirety. There are, nevertheless, a considerable number of experimental indications that the universal equilibrium theory holds for shear flows as well as for homogeneous isotropic fields to the extent that the Kolmogoroff spectrum has been measured within the boundary layers and the freejets.[10]

In the present section, we shall assume that the universal equilibrium theory holds at each point of the flowfield except in the laminar sublayer. This assumption implies that the rate of dissipation of the turbulence energy by the small "dissipation" wavenumbers is exactly equal to the energy depletion rate of the larger "energy-containing" wavenumbers *locally*. The rate of energy loss from the larger energy-containing wavenumbers is given[14] by $\beta\ \langle U_kU_k\rangle$ where $1/\beta$ is the characteristic time of the energy-containing eddies. Hence, we may write for the dissipation ϵ', with the aid of Eq. (17),

$$\epsilon' = (A/2\Lambda)\langle U_kU_k\rangle^{3/2} \tag{48}$$

In the analysis of Couette flow, Λ was set equal to $L/2$ in order not to further complicate the already formidable governing equations. However, various experimental data on shear flows indicate that the characteristic scales of the larger eddies are related to the local mean velocity gradients. Knowing that Λ must be finite at the edges of the shear flowfields ($y = L$), where the mean velocity gradient often approaches zero, we let, for two-dimensional flows,

$$\Lambda = L\Big/\left[\frac{\partial(u_o/u_{o\infty})}{\partial(y/L)} + 1\right] \tag{49}$$

where $u_{o\infty}$ is the reference average velocity with which the velocity is normalized.

With the constant A set equal to one, Eq. (48) is now determinate with Eq. (49). The dissipation functions for the present Couette flow, flat-plate boundary-layer flow, and the plane jet flow have been computed using Eq. (48). The computed results and the available corresponding experimental values of dissipation functions are shown in Fig. 4. The characteristic lengths L for the boundary layer and plane jet flows in Fig. 4 are the boundary-layer thickness and the half-width of the jet, respectively. The reference velocity $u_{o\infty}$ for the boundary layer is the freestream velocity, whereas that for the jet is the centerline velocity. The present computed values of $\langle U_kU_k\rangle$ have been employed in Eq. (48) for Couette flow, whereas the available experimental values of $\langle U_kU_k\rangle$ have been used for the boundary layer and jet flows.

Considering the discrepancy among the experimental values themselves, the agreements between Eq. (48), with Eq. (49), and the experimental values are quite satisfactory.

VI. Concluding Remarks

A simplified turbulence model has been constructed which describes the effect of various wavenumbers on the statistical behavior of fluid elements containing chemically reactive species. A modified Fokker-Planck equation has been derived and adapted to govern the distribution function of the fluid elements and, hence, of the chemical species. First, four-order moment equations were derived from the present theory, and they were shown to be identical up to a term to those derived from the Navier-Stokes equation, thus establishing the basic consistency of the present theory.

The theory was then employed, as a primary test, to analyze the turbulent Couette flow of a chemically inert fluid. In the analysis, it was shown that the present theory as applied to a chemically inert fluid is self-containing up to the dissipation function. An experimental value of the dissipation function for pipe flow, for lack of the same for Couette flow, was adapted and used in the analysis. A comparison of the present theoretical results with the available experimental data on Couette flow showed a satisfactory agreement between the two.

A simple relationship has been derived between the dissipation function and the other variables of the theory, based on the universal equilibrium theory, which would render the present theory completely self-containing for the chemically inert single-component flow. It was shown that this relationship predicts the dissipations for Couette, boundary layer, and plane jet flows quite satisfactorily.

In a high-Reynolds-number turbulence field, a statistical separation exists between the lower and the higher wavenumbers. Hence, there are two governing characteristic lengths or times of the flow. Furthermore, the higher wavenumbers are in statistical equilibrium. These facts enabled us to employ Brownian stochastics in describing the decay of the influence of the lower wavenumbers on the fluid elements. In Brownian stochastics, it is necessary to specify the characteristic time of the lower nonequilibrium wavenumbers only, since the characteristic time of the higher equilibrium wavenumbers is much shorter than that of the lower wavenumbers. With the assumption that the observable statistical properties of the fluid elements are entirely due to the lower wavenumbers, we were able to analyze the various properties of the flowfield, except the dissipation, in terms of the characteristic time of the lower wavenumbers only, in addition to the characteristic flow time.

The dissipation occurs at the higher wavenumbers. Therefore, the experimentally obtained value of the dissipation was employed in the solution of the Couette flow problem, or, equivalently, the experimental value of the characteristic time of the higher wavenumbers was employed. Subsequently, however, by assuming that the universal equilibrium theory holds locally in shear flow, we expressed the dissipation in terms of the characteristic time or length of the lower wavenumbers. Thus, we have completely described the turbulent flowfields in terms of the characteristic times of the lower wavenumbers and the flows.

The characteristic lengths (or times) of the lower wavenumbers in shear flows, except in those produced by the grids, for instance, are of the order of the characteristic lengths of the flows. We have, in Eq. (49), derived an expression for the characteristic length of the lower wavenumbers in terms of the characteristic length of the flow. Therefore, the turbulent shear flow problems as described by the present theory are now completely determinate when the flow is comprised of one chemically inert species. One should now be able to bring the concept of the lower-wavenumber controlled dissipation into Langevin's equation, Eq. (1), and improve the present theory. This will be included in a future paper.

References

[1] Proudian, A. P. and Feldman, S., "A New Model for Mixing and Fluctuations in a Turbulent Wake," *AIAA Journal*, Vol. 3, No. 4, April 1965, pp. 602–609.

[2] Lin, S. C., "A Bimodal Approximation for Reacting Turbulent Flows, I. Description of the Model," *AIAA Journal*, Vol. 4, No. 2, Feb. 1966, pp. 202–209.

[3] Lin, S. C., "A Bimodal Approximation for Reacting Turbulent Flows, II. Example of Quasi-One-Dimensional Wake Flow," *AIAA Journal*, Vol. 4, No. 2, Feb. 1966, pp. 210–216.

[4] Sutton, G. W., "Fluctuation Intensity of Passive Species in Turbulent Shear Flows," Research Report 254, Dec. 1966, Avco Everett Research Labs., Everett, Mass.

[5] Gibson, W. E., "Stochastic Model for Turbulent, Reacting Wakes," *AIAA Journal*, Vol. 4, No. 11, Nov. 1966, pp. 2001–2007.

[6] Harlow, F. H. and Nakayama, P. I., "Turbulence Transport Equations," *The Physics of Fluids*, Vol. 10, No. 11, Nov. 1967, pp. 2322–2332.

[7] Nee, V. W. and Kovasznay, S. G., "A Phenomenological Theory of Quasi-Parallel Turbulent Shear Flows," Rept. 1, 1967, Dept. of Mechanics, Johns Hopkins Univ., Baltimore, Md.

[8] Deissler, R. G., "Turbulent Heat Transfer and Temperature Fluctuations in a Field with Uniform Velocity and Temperature Gradients," *International Journal of Heat and Mass Transfer*, Vol. 6, 1963, pp. 257–270.

[9] Deissler, R. G., "Turbulence in the Presence of a Vertical Body Force and Temperature Gradient," *The Journal of Geophysical Research*, Vol. 67, 1962, pp. 3049–3062.

[10] Hinze, J. O., *Turbulence*, McGraw-Hill, New York, 1959.

[11] Hopf, E., "Statistical Hydrodynamics and Functional Calculus," *Journal of Rational Mechanical Analysis*, Vol. 1, 1952, pp. 87–123.

[12] Lundgren, T. S., "Distribution Functions in the Statistical Theory of Turbulence," *The Physics of Fluids*, Vol. 10, No. 5, May 1967, pp. 969–975.

[13] Chung, P. M., "A Simplified Description of Turbulent Chemically Reacting Flows," TR-1001(S2855-20)-5, 1967, Aerospace Corp., San Bernardino, Calif.

[14] Batchelor, G. K., *The Theory of Homogeneous Turbulence*, Cambridge University Press, Cambridge, England, 1960.

[15] Prigogine, I., *Non-Equilibrium Statistical Mechanics*, Interscience, New York, 1962, Chaps. 2 and 3.

[16] Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," *Review of Modern Physics*, Vol. 15, 1943, pp. 1–89.

[16] Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," *Review of Modern Physics*, Vol. 15, 1943, pp. 1–89.

[17] Yaglom, A. M., *Theory of Stationary Random Functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[18] Chapman, S. and Cowling, T. G., *Mathematical Theory of Nonuniform Gases*, Cambridge University Press, Cambridge, England, 1939.

[19] Liu, C. Y. and Lees, L., "Kinetic Theory Description of Plane Compressible Couette Flow," *Rarefied Gas Dynamics*, edited by L. Talbot, Academic Press, New York, 1961, pp. 391–428.

[20] Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1955.

[21] Robertson, J. M., "On Turbulent Plane-Couette Flow," *Proceedings of the Sixth Annual Conference on Fluid Mechanics*, Austin, Texas, 1959, pp. 169–182.

[22] Van Dyke, M. D., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1965.

[23] Reichardt, V. H., "Über die Geschwindigkeitsverteilung in einer Geradlinigen Turbulenten Couetteströmung," *Zeitscrift fur Angewandte Mathematik und Mechanik*, Vol. 36, 1956, pp. 26–29.

[24] Chung, P. M., "Chemically Reacting Nonequilibrium Boundary Layers," *Advances in Heat Transfer*, Vol. 2, Academic Press, New York, 1965, Chap. 2.

[25] Bradbury, L. J. S., "The Structure of a Self-Preserving Turbulent Plane Jet," *Journal of Fluid Mechanics*, Vol. 23, 1965, pp. 31–64.

[26] Heskestad, G., "Hot-Wire Measurements in a Plane Turbulent Jet," *Journal of Applied Mechanics*, Vol. 32, 1965, pp. 721–734.